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Probabilistic Teleportation via Entanglement

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Abstract With an arbitrary bi-particle entangled mixed state which is shared by Alice (the sender) and Bob (the receiver) acted as a quantum channel, at first, a teleportation protocol that Alice successfully transmits an unknown mixed state to Bob based on a positive operator-valued measurement (POVM) is presented. The upper bound of probability to teleport successfully an unknown mixed state is then investigated, and conclude that it completely depends on the entanglement degree of the bi-particle entangled mixed state as a resource.

Keywords Probabilistic teleportation · Mixed state · Entanglement · Optimal probability · Measurement

1 Introduction

The no-cloning theorem indicates that a quantum state cannot be cloned exactly [20]. However, quantum copying is necessary in quantum computation and quantum information [2, 6, 7]. To a large extent, the teleportation protocol that is one of the most profound discovery of quantum information theory plays an important role in copying quantum information [4]. The problem of quantum teleportation is whether there exists a physical device and a key (or a set of keys), and these kind of physical ones can accomplish a task that is a quantum state attached to a sender (Alice) will completely be transmitted and a receiver (Bob) can reconstruct the state sent. In [4] the teleportation is to divide the information encoded in the pure state into two parts, classical and quantum, and send them through different channels, a classical channel and an EPR (Einstein-Podolsky-Rosen) channel. The classical channel is nothing but a simple correspondence between the sender and the receiver, and the EPR channel is constructed by using a Bell state. The results are extended to the N-dimensional quantum teleportation [5, 11]. In these schemes the maximally entangled pure state acts as an ideal noiseless quantum channel. It is easy to prove that if and only if teleportation can

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success faithfully via the maximally entangled state. It means that the receiver can recover the pure state sent with probability 1.

1.1 Previous Results

In realistic situation, a maximally entangled pure state cannot be kept for a long time due to the decoherence. So people have to consider the case when Alice and Bob share a non-maximally (partially) entangled state. In this case, Bob cannot faithfully reconstruct the state which is sent by Alice. In non-deterministic event, to indicate the success of the measurement, an extra single bit has to be sent to the receiver together with the two-bit information via a classical channel. So probabilistic (non-deterministic) teleportation means that Bob recover the state sent with probability $p(< 1)$. Of course, the maximal probability of teleportation is regarded as the most important index of non-deterministic teleportation process. Mor and Horodecki [15] proposed a probabilistic teleportation protocol for a quantum state in two-dimensional Hilbert space for a partially entangled quantum channel. They employed a POVM as the joint measurement at the sending station. When the measurement is successful by a random chance, the initial unknown quantum state is teleported perfectly. Bandyopadhyay [2] and Li et al. [13] have also proposed protocols to implement probabilistic teleportation in two-dimensional Hilbert space. In their scheme, Bandyopadhyay used a combination of orthogonal Greenberger-Horne-Zeilinger measurements and POVM's, while Li et al. took general transformations at the receiving station leaving the measurement orthogonal.

In contrast, for a deterministic event, the teleportation loses its quantum characteristics. In this case, Bob can not reconstruct the state sent. Since the overall fidelity for approximate teleportation is less than that for the standard teleportation, the fidelity between state reconstructed and state sent measures whether an approximate (deterministic) teleportation protocol is perfect or not. Son et al. [18] formulated an approximate teleportation in N-dimensional Hilbert space for partially entangled quantum channel, utilizing rank-one POVM for the joint measurement at the sending station and unitary transformation at the receiving station. The authors evaluated the teleportation fidelity and the probability of successful teleportation. In [1], the authors considered the maximal teleportation fidelity via an entangled mixed state by means of a quantum operation. Also other teleportation protocols were considered similarly [9, 14].

1.2 Motivation

However, Alice and Bob perhaps share an entangled mixed state which is regarded as a quantum channel in realistic situation due to the decoherence. Generally speaking, teleportation using a mixed state as an entangled resource is equivalent to a noisy quantum channel. The situation that we wish to tackle here is when Alice and Bob share a mixed state as a resource. It is well known whether teleportation can be accomplished depends on what resource they share. Similar to partially entangled quantum channel, the teleportation using a mixed state as a resource is non-faithful. So it is valuable to investigate the fidelity and the successful probability of teleportation. In addition, quantum teleportation does not prohibit from transmitting an unknown mixed state. In this paper, we consider a probabilistic teleportation protocol that Alice transmits an unknown mixed state to Bob via an entangled mixed state based on a POVM. The performance to achieve this protocol comprises Alice's operation and Bob's. In other words, Alice performs a POVM measurement on her systems $\mathcal{H}_1 \otimes \mathcal{H}_2$, and then informs Bob her results of measurement with a classical channel. In the next step, Bob utilizes some unitary operators on his system $\mathcal{H}_3$ in order to recover state sent with a probability p (≤ 1) according to Alice's measurement results. However, this protocol does not simply generalize the case of pure state, since the non-determinacy of mixed state results in that it is much more difficult to deal with mixed states than pure ones. The marked differences in this protocol is that Bob uses an unitary operator U (see (9) and (14)). Investigated the maximal successful probability of teleportation from mathematical viewpoint, we find that such process to construct an unitary operator indeed ensure that this protocol is an optimal probabilistic teleportation based on a POVM. It also means that our scheme is constructive and credible.

2 Teleportation with Entangled Pure State

We first summarize briefly some basic concepts and results that are needed for our further treatment. An ensemble of pure states is characterized by a finite set of positive numbers p_i ($\sum_i p_i = 1$) and by a corresponding set of normalized vectors $|\psi_i\rangle$ of the Hilbert space $\mathcal{H}$, which usually is represented by $\{p_i, |\psi_i\rangle\}$ [3]. The density operator ρ (a trace one, positive semidefinite operator) associated to $\{p_i, |\psi_i\rangle\}$ is defined as:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad \sum_i p_i = 1, \quad p_i \geq 0 \tag{1}$$

In the quantum information field, a density operator is also called a mixed state. Obviously, a pure state is a special case for $i = 1$ in (1). However, it is more difficult to tackle a mixed state than a pure one, as a mixed state is non-determinate.

Definition 2.1 [19] A channel converting quantum systems with Hilbert space $\mathcal{H}_{\text{in}}$ into systems with Hilbert space $\mathcal{H}_{\text{out}}$ is a linear operator $\varepsilon : \mathcal{B}(\mathcal{H}_{\text{out}}) \to \mathcal{B}(\mathcal{H}_{\text{in}})$, which is completely positive and normalized as $\varepsilon(I) = I$.

According to Kraus representation [12], a quantum operation defined as a completely positive and trace-preserving linear mapping on the state space $\mathcal{H}$ can be denoted by

$$\varepsilon(\rho) = \sum_{k=1}^{N} A_k \rho A_k^\dagger \tag{2}$$

where $A_k^\dagger$ is the conjugate transpose of A_k. To guarantee the trace-preserving property, the following completeness condition $\sum_{k=1}^{N} A_k^\dagger A_k = I$ is also required.

Definition 2.2 [5] Let $X|k\rangle = |k+1\rangle$, $Z|k\rangle = \omega^k|k\rangle$, where $\omega = e^{-\frac{2\pi i}{N}}$. $\sigma_{mn} = X^m Z^n$ ($0 \leq m, n \leq N-1$) are called generalized Pauli matrix of N-dimensional quantum system $\mathcal{H}$.

Definition 2.3 [5] We refer to the following as generalized Bell states of $N \times N$-dimensional quantum system $\mathcal{H}^{\otimes 2}$

$$|B_{mn}\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i k n}{N}} |k\rangle|k+m\rangle \tag{3}$$

It is easy to find that the set of $\{|B_{mn}\rangle\}_{m,n=0}^{N-1}$ constitutes an orthonormal basis of Hilbert space $\mathcal{H}^{\otimes 2}$. In addition, we can rewrite $|B_{mn}\rangle = (I \otimes \sigma_{mn}^*)|I\rangle$ with $|I\rangle = \frac{1}{\sqrt{N}} \sum_i |ii\rangle$.

A quantum state $|\psi\rangle$ is a vector in the N-dimensional Hilbert space $\mathcal{H}$ denoted as

$$|\psi\rangle = \sum_{i=1}^{N} b_i |i\rangle, \quad b_i \in \mathbb{C}, \quad \sum_{i=1}^{N} |b_i|^2 = 1 \tag{4}$$

A POVM for the joint measurement on the quantum system $\mathcal{H}^{\otimes 2}$ is defined as a partition of unity by the positive operators, which are nonorthogonal in general. A set of POVM operators $\{M_k\}$ with $n(\geq N^2)$ outcomes satisfy the measurement conditions of positivity and completeness, $\sum_{k=1}^{n} M_k = I$ [16].

Suppose Alice possess two N dimensional quantum systems while Bob has one. Alice's systems will be labeled 1 and 2 while Bob's will be labeled 3. We assume that the quantum channel, which Alice and Bob share, is prepared with a pure entangled pair of particles 2 and 3 in the state $|\Phi\rangle$. According to Schmidt decomposition [17], $|\Phi\rangle$ can be denoted as

$$|\Phi\rangle = \sum_{i=1}^{N} \lambda_i |ii\rangle, \quad \sum_{i=1}^{N} \lambda_i^2 = 1, \quad \lambda_i \geq 0 \tag{5}$$

Let $A = \mathrm{diag}(\lambda_1 \lambda_2 \cdots \lambda_N)$, then $|\Phi\rangle$ can be rewritten as $|\Phi\rangle = (I \otimes A)|I\rangle = (A^T \otimes I)|I\rangle$.

Obviously, the total quantum state for the unknown particle 1 and the entangled pair 2 and 3 is given by the tensor product of the unknown state $|\psi\rangle$ and the quantum channel state $|\Phi\rangle$,

$$|\psi\rangle \otimes |\Phi\rangle = \sum_{i,j=1}^{N} \lambda_i b_j |jii\rangle \tag{6}$$

Similar to [18], we define the basis set of entangled states obtained by using the same local unitary operators $\sigma_{mn}^{\dagger}$ on the state of the quantum channel $|\Phi\rangle$,

$$|\phi_{mn}\rangle = (\sigma_{mn}^{\dagger} \otimes I)|\Phi\rangle, \quad m, n = 1, 2, \ldots, N \tag{7}$$

The basis states $|\phi_{mn}\rangle$ are not necessarily orthogonal. Only when the channel is maximally entangled, does the set of the basis states $|\phi_k\rangle$ represent the Von Neumann orthogonal measurement. The total state $|\psi\rangle \otimes |\Phi\rangle$ can now be rewritten as

$$|\psi\rangle \otimes |\Phi\rangle = \frac{1}{N^2} \sum_{m,n=1}^{N} |\phi_{mn}\rangle \otimes \sigma_{mn}|\psi\rangle \tag{8}$$

Now we construct a probabilistic teleportation scheme that Alice transmits an unknown state $|\psi\rangle$ to Bob via a partly entangled quantum channel $|\Phi\rangle$ based on a POVM. First of all, Alice performs a joint POVM which can discern the nonorthogonal states in the right hand of (8) on her particles 1 and 2, and informs Bob her measurement results using a classical channel. without loss of the generality, we assume that $|\phi_{mn}\rangle$ is one of the possible options of Alice's measurement results. In this case, Bob's system $\mathcal{H}_3$ gets into the state $|\psi\rangle \to \frac{1}{\sqrt{N}} A\sigma_{mn}|\psi\rangle$. To recover the unknown state $|\psi\rangle$, we introduce an auxiliary system $\mathcal{H}_{\mathrm{aux}}(\dim\mathcal{H}_{\mathrm{aux}} = N)$ with the original state $|0\rangle_{\mathrm{aux}}$ to carry out an evolution. In the next step, Bob first utilizes a collective unitary operator U on systems $\mathcal{H}_3 \otimes \mathcal{H}_{\mathrm{aux}}$ expressed as the following

$$U = \mathrm{diag}(U_1 U_2 \cdots U_N) \tag{9}$$

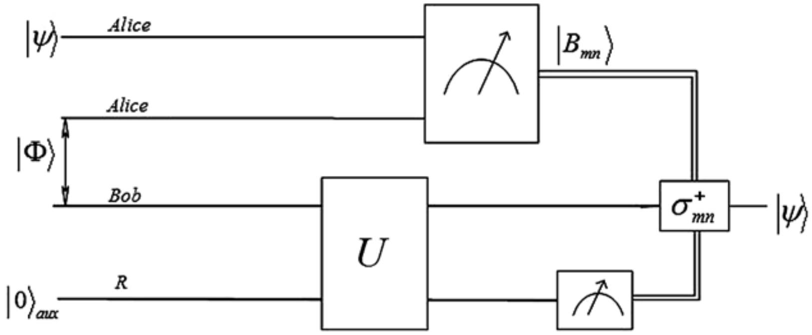

Fig. 1 Where the meter denotes a quantum measurement. In addition to bi-line means that it transmits some classical bits

where the block matrix

$$U_k = e^{-\frac{2\pi ikn}{N}} \begin{pmatrix} \frac{\lambda_{\min}}{\lambda_k} & -\sqrt{1-(\frac{\lambda_{\min}}{\lambda_k})^2} & \\ \sqrt{1-(\frac{\lambda_{\min}}{\lambda_k})^2} & \frac{\lambda_{\min}}{\lambda_k} & \\ & & I \end{pmatrix}_{N\times N}$$

$\lambda_{\min} = \min_k \lambda_k$ and I is identity.

The systems $\mathcal{H}_3 \otimes \mathcal{H}_{\text{aux}}$ become $\frac{1}{\sqrt{N}} U(A\sigma_{mn}|\psi\rangle)|0\rangle_{\text{aux}}$. Furthermore, $\frac{1}{\sqrt{N}} U(A\sigma_{mn}|\psi\rangle)|0\rangle_{\text{aux}}$ can be rewritten as

$$\frac{1}{\sqrt{N}}(\lambda_{\min}\sigma_{mn}|\psi\rangle)|0\rangle_{\text{aux}} + \sum_{j=1}^{N} |\varphi_j\rangle|j\rangle_{\text{aux}} \tag{10}$$

where $|\varphi_j\rangle$ denotes some quantum states, which can be ignored in this case. Then Bob performs a measurement on the auxiliary system $\mathcal{H}_{\text{aux}}$. If the measurement result is $|0\rangle_{\text{aux}}$, the teleportation is successfully accessed, otherwise means that the teleportation fails. In this case, Bob obtains the state $\sigma^\dagger_{mn}|\psi\rangle$ with a probability $\frac{\lambda^2_{\min}}{N}$. Finally, Bob recovers the unknown state $|\psi\rangle$ sent by Alice after performing an unitary operator $\sigma^\dagger_{mn}$ on his system $\mathcal{H}_3$ according to Alice's measurement results.

This protocol can be depicted by the quantum circuitry in Fig. 1.

We assert the fact that such an unitary operator in (9) can ensure that the probability of teleporting successfully will be maximal. Without loss of the generality, suppose Bob uses an arbitrary unitary operator V on systems $\mathcal{H}_3 \otimes \mathcal{H}_{\text{aux}}$. It is not difficult to obtain a channel $\varepsilon : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_{\text{aux}}) \to \mathcal{B}(\mathcal{H}_3 \otimes \mathcal{H}_{\text{aux}})$ expressed as

$$\varepsilon(|\psi\rangle|0\rangle_{\text{aux}}\langle 0|\langle\psi|) = \frac{1}{N}\sum_{mn} V(A\sigma^\dagger_{mn}|\psi\rangle)|0\rangle_{\text{aux}}\langle 0|(\langle\psi|\sigma_{mn}A^\dagger)V^\dagger \tag{11}$$

We assume that Bob obtains the measurement result $|0\rangle_{\text{aux}}$ with a probability $p(0)$. Now we come to compute the probability $p(0)$. Since V can be defined by ${}_{\text{aux}}\langle k|V|0\rangle_{\text{aux}} = E_k$ according to Kraus representation [12], $p(0)$ can be expressed as $\frac{1}{N}\langle A\sigma^\dagger_{mn}\psi|E^\dagger_0 E_0 A\sigma_{mn}\psi\rangle$. Next we consider the upper bound of $p(0)$. Since $|\psi\rangle$ is unknown, $p(0)_{\max}$ can be rewritten as

$$p(0)_{\max} = \frac{1}{N} \min_{\varphi \in \mathcal{H}} \langle A\sigma_{mn}^{\dagger}\psi | E_0^{\dagger} E_0 A\sigma_{mn}\psi \rangle \tag{12}$$

For brevity, we shall write A' instead of $(E_0)^{\dagger} E_0$. Since a diagonal matrix is easier to handle than the non-diagonal ones, we now diagonalize $A' = U_1 \Lambda' U_1^{\dagger}$ with $\Lambda' = \mathrm{diag}(\lambda'_1, \ldots, \lambda'_N)$. Equation (12) can be rewritten as

$$\min_{\varphi \in \mathcal{H}} \langle A\varphi | A' A\varphi \rangle = \min_{\varphi \in \mathcal{H}} \langle U_1^{\dagger} A\varphi | \Lambda' U_1^{\dagger} A\varphi \rangle = \min_{\varphi \in \mathcal{H}} \langle A\varphi | \Lambda' A\varphi \rangle \tag{13}$$

Proposition 2.4 $\min_{x_i} \sum_{i=1}^{n} a_i x_i = \min_i a_i$, *where* $\sum_i a_i = 1$, $\sum_i x_i = 1$ *and* $a_i, x_i \geq 0$.

Proof We prove the case of $n = 3$. The other cases can be deduced similarly. $\min_{x_i} \sum_{i=1}^{n} a_i x_i = \min(a_3 + (a_1 - a_3)x_1 + (a_2 - a_3)x_2) = \begin{cases} \min\{a_1, a_2\} & a_3 \geq a_1, a_2 \\ a_3 & a_3 < a_1, a_2 \end{cases}$. So $\min_{x_i} \sum_{i=1}^{n} a_i x_i = \min_i a_i$ holds. □

Using Proposition 2.4, the right hand of (12) becomes $\min_i \lambda'_i \lambda_{ii}^2 \leq \min_i \lambda_{ii}^2$. It is easy to find that if A' can be diagonalized as $V' A' V'^{\dagger} = \mathrm{diag}(\frac{a_{\min}}{\lambda_1}, \ldots, 1, \ldots, \frac{a_{\min}}{\lambda_N})$, then the "=" holds which mentioned previously, where the i-th eigenvalue of A' equals 1.

Remark 1 In fact (12) holds all the same if A is non-diagonal.

Remark 2 The condition that "=" holds gives an approach to construct an unitary operator such that the successful probability of teleportation will be maximal. We construct U in (9) in term of this method.

In a word, let Alice and Bob share an entangled pure state $|\Phi\rangle = (I \otimes A)|I\rangle$ with $A = (a_{ii})_{N \times N}$, $\sum_{i=1}^{N} a_{ii}^2 = 1$, $a_{ii} \geq 0$. For an arbitrary unknown pure state $|\psi\rangle$, if Alice and Bob utilize the process that mentioned above (see Fig. 1).

Theorem 2.5 *The probability that Alice successfully teleports* $|\psi\rangle$ *to Bob is less than or equal to* $Na_{\min}^2$ *with* $a_{\min} = \min_j \{a_{jj}\}$. *Furthermore, if we construct an unitary operator as what mentioned previously, then the successful probability meets* $Na_{\min}^2$.

Proof As the argument above, Alice obtains the outcomes of measurement $|\phi_{mn}\rangle$ with the same probability $\frac{1}{N^2}$. Bob obtains the measurement result $|0\rangle_{\mathrm{aux}}$ with probability $p(0)$. Hence, the total optimal successful probability of teleportation is $p = N^2 p(0)_{\max} \leq Na_{\min}^2$. □

3 In Case of Entangled Mixed State as Resource

As the mentioned above, teleportation using entangled mixed state as a resource is equivalent to having a noisy quantum channel. In fact, the teleportation protocol does not prohibit to transform an unknown mixed state. What we are interested in is when Alice teleports an unknown mixed state via an arbitrary entangled mixed state as a resource. Let Alice and Bob share an arbitrary bi-particle entangled mixed state ρ' acted as a resource.

The entanglement of formation for the mixed state ρ is defined as the average entanglement of the pure states of the decomposition, minimized over all possible decompositions of ρ, $E(\rho)=\min_{\{p_i,|\psi_i\rangle\}}\sum_i p_i E(|\psi_i\rangle)$ [8]. There always is a pure ensemble $\{p_k;\ |\psi_k\rangle\}$ such that $E(\rho)=\sum_k p_k E(|\psi_k\rangle)$. ρ' hence can be written as $\rho'=\sum_k p_k|\psi_k\rangle\langle\psi_k|$ with $|\psi_k\rangle=\sum_i a_{ii}^k|i_k i_k\rangle$. Furthermore, ρ' can be parameterized by

$$\rho'=\frac{1}{N}\sum_k p_k(I\otimes A_k)|I\rangle\langle I|(I\otimes A_k)^\dagger \tag{14}$$

where $A_k=\mathrm{diag}(a_{11}^k a_{22}^k\cdots a_{NN}^k)$. We can choose the state $|\psi_k^0\rangle$ such that $E(|\psi_k^0\rangle)=\min_k\{E(|\psi_k\rangle)\}$. Let $|\phi'_{mn}\rangle=(\sigma_{mn}^\dagger\otimes I)|\psi_k^0\rangle$. Alice intends to transmit an unknown mixed state ρ. Similar to Sect. 2, Alice first performs a joint POVM $\{|\phi'_{mn}\rangle\langle\phi'_{mn}|\}_{m,n=0}^{N-1}$ on her systems $\mathcal{H}_1\otimes\mathcal{H}_2$, and informs Bob her measurement results. Next Bob utilizes a collective unitary operator U_k depicted the following on systems $\mathcal{H}_3\otimes\mathcal{H}_{\mathrm{aux}}$.

$$U_k=\mathrm{diag}(U_1^k U_2^k\cdots U_N^k) \tag{15}$$

where the block matrix

$$U_j^k=e^{-\frac{2\pi ijn}{N}}\begin{pmatrix}\frac{a_{\min}}{a_{jj}^k} & -\sqrt{1-(\frac{a_{\min}}{a_{jj}^k})^2} & \\ \sqrt{1-(\frac{a_{\min}}{a_{jj}^k})^2} & \frac{a_{\min}}{a_{jj}^k} & \\ & & I\end{pmatrix}_{N\times N}$$

$a_{\min}=\min_{j,k}\{a_{jj}^k\}$ and I is identity. Then Bob performs a measurement on the auxiliary system $\mathcal{H}_{\mathrm{aux}}$. If the measurement result is $|0\rangle_{\mathrm{aux}}$, the teleportation is successfully accessed, otherwise means that the teleportation fails. Finally, Bob recovers the unknown state ρ sent by Alice with a probability $\frac{a_{\min}p_k}{N}$, after performing an unitary operator σ_{mn} on his system $\mathcal{H}_3$ according to Alice' s measurement results.

We assert the fact that such an unitary operator in (15) can ensure that the probability to teleport successfully will be maximal. Without loss of the generality, suppose Bob uses an arbitrary unitary operator V_k on systems $\mathcal{H}_3\otimes\mathcal{H}_{\mathrm{aux}}$. It is not difficult to obtain a channel $\varepsilon:\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_{\mathrm{aux}})\to\mathcal{B}(\mathcal{H}_3\otimes\mathcal{H}_{\mathrm{aux}})$ expressed as

$$\varepsilon(\rho\otimes|0\rangle_{\mathrm{aux}}\langle 0|)=\frac{1}{N}\sum_{mnk}p_k V_k(A_k\sigma_{mn}\rho\sigma_{mn}^\dagger A_k^\dagger)\otimes|0\rangle_{\mathrm{aux}}\langle 0|V_k^\dagger \tag{16}$$

the trace-preserving property of the quantum channel can be proved by

$$\begin{aligned}\mathrm{tr}(\varepsilon(\rho\otimes|0\rangle_{\mathrm{aux}}\langle 0|)) &= \frac{1}{N}\sum_{mnk}p_k\mathrm{tr}(V_k(A_k\sigma_{mn}\rho\sigma_{mn}^\dagger A_k^\dagger)\otimes|0\rangle_{\mathrm{aux}}\langle 0|V_k^\dagger)\\ &= \frac{1}{N}\sum_{mnk}p_k\mathrm{tr}((\sigma_{mn}A_k^\dagger A_k\sigma_{mn}^\dagger\rho)\otimes|0\rangle_{\mathrm{aux}}\langle 0|)\end{aligned} \tag{17}$$

Using the identity $\sum_{mn}\sigma_{mn}A\sigma_{mn}^\dagger=N\mathrm{tr}(A)I_N$ for any $N\times N$ matrix A, (17) can be rewritten as $\mathrm{tr}(\varepsilon(\rho\otimes|0\rangle_{\mathrm{aux}}\langle 0|))=\sum_k p_k\mathrm{tr}(A_k^\dagger A_k)\times\mathrm{tr}(\rho\otimes|0\rangle_{\mathrm{aux}}\langle 0|)=1$, where we utilize $\mathrm{tr}(\rho\otimes|0\rangle_{\mathrm{aux}}\langle 0|)=1$.

We assume that Bob obtains the measurement result $|0\rangle_{\text{aux}}$ with probability $q(0)$. Since V_k can be defined by ${}_{\text{aux}}\langle l|V_k|0\rangle_{\text{aux}} = E_l^k$ according to Kraus representation, it is not difficult to see that $q(0) = \frac{1}{N}\sum_k p_k \text{tr}(E_0^k A_k \sigma_{mn}^\dagger \rho \sigma_{mn} A_k^\dagger (E_0^k)^\dagger)$. Next we investigate the upper bound of $q(0)$. Since ρ is unknown, $q(0)_{\max}$ can be rewritten as

$$q(0)_{\max} = \min_{\rho\in\mathcal{H}} \frac{1}{N}\sum_k p_k \text{tr}(E_0^k A_k \sigma_{mn}^\dagger \rho \sigma_{mn} A_k^\dagger (E_0^k)^\dagger) \tag{18}$$

Let $\rho'' = \sigma_{mn}^\dagger \rho \sigma_{mn}$. Then (18) can be rewritten as

$$p(0)_{\max} = \min_{\rho''\in\mathcal{H}} \frac{1}{N}\sum_k p_k \text{tr}(E_0^k A_k \rho'' A_k^\dagger (E_0^k)^\dagger) = \min_{\rho\in\mathcal{H}} \frac{1}{N}\sum_k p_k \text{tr}(E_0^k A_k \rho A_k^\dagger (E_0^k)^\dagger) \tag{19}$$

Diagonalizing $(E_0^k)^\dagger E_0^k = V_k' \Lambda_k V_k'^\dagger$ with $\Lambda_k = \text{diag}(\lambda_{k1}\lambda_{k2}\cdots\lambda_{kN}')$, then (19) becomes as

$$q(0)_{\max} = \min_{\rho\in\mathcal{H}} \frac{1}{N}\sum_k p_k \text{tr}(V_k' \Lambda_k V_k'^\dagger A_k \rho A_k^\dagger) \tag{20}$$

Using Cauchy-Schwarz inequality, $\text{tr}((V_k'\Lambda_k V_k'^\dagger)^\dagger A_k\rho A_k^\dagger) \le \text{tr}\Lambda_k \sqrt{\text{tr}((A_k\rho A_k^\dagger)^2)}$. Using $\sqrt{\text{tr}A^2} \le \text{tr}A (A \ge 0)$, then $q(0)_{\max} \le \frac{1}{N} p_k \min_{i,k}\{(a_{ii}^k)^2\} = \frac{1}{N} p_k a_{\min}^2$, where $a_{\min}^2 = \min_{i,k}\{(a_{ii}^k)^2\}$.

Theorem 3.1 *Let Alice and Bob share an entangled mixed state $\rho' = \sum_k p_k|\varphi_k\rangle\langle\varphi_k|$ with $|\varphi_k\rangle = \sum_i a_{ii}^k |i_k i_k\rangle$. For an arbitrary unknown mixed state ρ, if Alice and Bob utilize the teleportation protocol that mentioned above, then the successful probability that Alice teleports ρ to Bob is less than or equal to $Np_k a_{\min}^2$ with $a_{\min} = \min_{i,k}\{a_{ii}^k\}$.*

Proof The proof is similar to Theorem 2.5. □

For any entangled bi-particle pure state, it is well known that any measurement of entanglement for them can be written as a function of x, that is, $E(\phi) = f(x)$, where x denotes N times of the minimal Schmidt coefficient [16] of $|\phi\rangle$. The function $f(x)$ must satisfy the following conditions [10]:

(i) Monotonicity: $\forall x_1 \ge x_2$, $f(x_1) \ge f(x_2)$
(ii) Concavity: $f(\sum p_i x_i) \ge \sum p_i f(x_i)$
(iii) Normalization: $f(0) = 0$ and $f(1) = 1$

We can prove the following lemma with the help of condition (ii) and (iii):

Lemma 3.2 [10] $f(x) \ge x$.

Comparing Lemma 3.2 with Theorem 2.5, we can find surprisedly that the entanglement degree of bi-particle pure state can be considered as the width of the quantum channel, because $Na_{\min}^2 \le Na_{\min}(= x) \le f(x)$ holds by Lemma 3.2. Then the amount of transmitted quantum information is determined by the low bound of quantum channel.

Let x_i denote N times of the minimal Schmidt coefficient of $|\varphi_i\rangle$. Then we have

$$E(\rho) = \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i f(x_i) \geq \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i f(x_i^{\min})$$
$$\geq f(x_i^{\min}) \geq N a_{\min} \geq N a_{\min}^2 \geq N p_k a_{\min}^2 \geq q(0)_{\max} \quad (21)$$

where $x_i^{\min} = \min_i x_i$ and $a_{\min} = \min_{i,j}\{a_{ii}^j\}$. Similar to what mentioned above, the entanglement degree of mixed state which is shared by Alice and Bob determines the upper bound of probability of successful teleportation.

4 Conclusions

We give a protocol that the sender teleports an unknown mixed state to the receiver with an arbitrary mixed entangled state as a resource based on a POVM. This protocol will be more useful in realistic situation, since pure state can not stay for a long time due to drcoherence. As this protocol is non-deterministic, it is important to consider the successful probability of transmission. After investigating the upper bound of successful probability, we find that it is the minimal eigenvalue of the mixed entangled state that determines the capable of quantum channel to transmit an unknown mixed state. And only when Bob performs an unitary operation which is constructed in the way of Remark 2, the successful probability of teleportation is optimal. In addition, the successful probability of teleportation with the mixed entangled state as a resource will not better than with the pure entangled state. This result is right from out intuition and proof.

Of course, we are interested in whether the probability of successful teleportation can be increased in the protocol that appends some redundant particle.

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